FINITELY PRESENTED LATTICE-ORDERED ABELIAN GROUPS WITH ORDER-UNIT

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Abstract. Let G be an ℓ-group (which is short for "lattice-ordered abelian group"). Baker and Beynon proved that G is finitely presented iff it is finitely generated and projective. In the category $\mathcal{U}$ of *unital* ℓ-groups—those ℓ-groups having a distinguished order-unit u—only the $(\Leftarrow)$-direction holds in general. Morphisms in $\mathcal{U}$ are *unital ℓ-homomorphisms,* i.e., homomorphisms that preserve the order-unit and the lattice structure. We show that a unital ℓ-group (G, u) is finitely presented iff it has a basis, i.e., G is generated by an abstract Schauder basis over its maximal spectral space. Thus every finitely generated projective unital ℓ-group has a basis $\mathcal{B}$. As a partial converse, a large class of projectives is constructed from bases satisfying $\bigwedge \mathcal{B} \neq 0$. Without using the Effros-Handelman-Shen theorem, we finally show that the bases of any finitely presented unital ℓ-group (G, u) provide a direct system of simplicial groups with 1-1 positive unital homomorphisms, whose limit is (G, u).

1. Introduction

We refer to [4], [10] and [13] for background on ℓ-groups. A unital ℓ-group (G, u) is an abelian group G equipped with a translation-invariant lattice-order and a distinguished *order-unit* u, i.e., an element whose positive integer multiples eventually dominate each element of G. Unital ℓ-groups are a modern mathematization of the time-honored euclidean magnitudes with an archimedean unit (see [17]). By [19, Theorem 3.9], the category $\mathcal{U}$ of unital ℓ-groups is equivalent to the equational class of MV-algebras. Thus, while the archimedean property of order-units is not definable in first-order logic, $\mathcal{U}$ is endowed with all typical properties of equational classes: in particular, $\mathcal{U}$ has subalgebras, quotients and products, which in general are not cartesian products, [5].

Finitely presented ℓ-groups (with or without unit) are an active topic of current research, because they have a basic proteiform reality, as computable algebraic structures, rational polyhedra, fans, finitely axiomatizable theories in many-valued logic, and finitely presented AF C* algebras whose Murray-von Neumann order of projections is a lattice. See [11, 20, 18, 16, 21].

Morphisms in the category of ℓ-groups are lattice-preserving homomorphisms. Morphisms in the category of unital ℓ-groups are also required to preserve order-units. A unital ℓ-group (G, u) is *projective* if whenever $\psi\colon (A, a) \to (B, b)$ is a

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surjective morphism and $\phi\colon (G,u) \to (B,b)$ is a morphism, there is a morphism $\theta\colon (G,u) \to (A,a)$ such that $\phi = \psi \circ \theta$. For ℓ-groups, Baker [1] and Beynon [2], [3, Theorem 3.1] (also see [10, Corollary 5.2.2]) gave the following characterization: *An ℓ-group G is finitely generated projective iff it is finitely presented.* For unital ℓ-groups the ($\Rightarrow$)-direction holds ([21, Proposition 5]). The converse direction fails in general.

Schauder bases provide the main tool to prove that an ℓ-group is finitely generated projective iff it is presented by a word in the pure language of lattices, without resorting to the group structure, [16]. This strengthens the characterization by Baker-Beynon mentioned above, where lattice-group words are used, and paves the way to a full understanding of the sharp differences between measure theory in unital and in non-unital ℓ-groups, [21].

For a geometric investigation of finitely presented unital ℓ-groups, in [18] the notion of *b*asis (see Definition 2.1) was introduced as a purely algebraic counterpart of Schauder bases. Specifically, in [18, Theorem 4.5] it is proved that an *archimedean* unital ℓ-group (G,u) is finitely presented iff it has a basis. The archimedean condition means that G is isomorphic to an ℓ-group of real-valued functions defined on some set X. In Theorem 3.1 we will prove that the archimedean assumption can be dropped, thus obtaining a characterization of finitely presented unital ℓ-groups that does not mention free objects and their universal property.

As a corollary, every finitely generated projective unital ℓ-group has a basis. In Section 4 we will prove a partial converse, yielding a method to construct large classes of projective unital ℓ-groups.

With reference to [9] and [12], the underlying dimension group of (G,u) will be considered in the final section. In Theorem 5.3 it is proved that if (G,u) has a basis then its bases provide a direct system of simplicial groups with 1-1 positive unital homomorphisms, whose limit is (G,u). Thus the Effros-Handelman-Shen representation theorem [6], Grillet's theorem [15, 2.1], and Marra's theorem [17] have a very simple proof for any such (G,u).

2. Preliminaries

2.1. **Unital ℓ-groups and bases.** A *lattice-ordered abelian group* (*ℓ-group*) is a structure $(G,+,-,0,\vee,\wedge)$ such that $(G,+,-,0)$ is an abelian group, $(G,\vee,\wedge)$ is a lattice, and $x+(y\vee z) = (x+y)\vee(x+z)$ for all $x,y,z\in G$. An *order-unit* in G is an element $u \in G$ with the property that for every $g\in G$ there is $n\in\{1,2,3,\ldots\}$ such that $g \le nu$. A *unital ℓ-group* (G,u) is an ℓ-group G with a distinguished order-unit u.

A map $h\colon (G,u)\to(G',u')$ is said to be a *unital ℓ-homomorphism* if it preserves the lattice as well as the group structure, and $h(u) = u'$. By an *ideal* $\mathfrak{i}$ of a unital ℓ-group (G,u) we mean the kernel of a unital ℓ-homomorphism of (G,u). We denote by $\mathrm{MaxSpec}(G,u)$ the set of maximal ideals of (G,u) equipped with the *spectral* topology, [4, §10]: a basis of closed sets for $\mathrm{MaxSpec}(G)$ is given by sets of the form $\{\mathfrak{p}\in \mathrm{MaxSpec}(G)\mid a\in\mathfrak{p}\}$, where a ranges over all elements of G. Since G has an order-unit, $\mathrm{MaxSpec}(G)$ is a nonempty compact Hausdorff space, [4, 10.2.2].

Definition 2.1. Let (G,u) be a unital ℓ-group. A *basis* of (G,u) is a set $\mathcal{B} = \{b_1,\ldots,b_n\}$ of nonzero elements of the positive cone $G^+ = \{g\in G\mid g\ge 0\}$ such that

(i) $\mathcal{B}$ generates G;
(ii) for each $k = 1, 2, \ldots$ and k-element subset C of $\mathcal{B}$ with $0 \neq \bigwedge\{b \mid b \in C\}$, the set $\{\mathfrak{m} \in \mathrm{MaxSpec}(G) \mid \mathfrak{m} \supseteq \mathcal{B} \setminus C\}$ is homeomorphic to a $(k-1)$-simplex;
(iii) there are integers $1 \leq m_1, \ldots, m_n$ such that $\sum_{i=1}^{n} m_i b_i = u$.

This is an equivalent simplified reformulation of [18, Definition 4.3]. From (ii)-(iii) it follows that the *multiplicity* m_i of each $b_i \in \mathcal{B}$ is uniquely determined.

For $n = 1, 2, \ldots$ we let $\mathcal{M}_n$ denote the unital ℓ-group of all continuous functions $f\colon [0,1]^n \to \mathbb{R}$ having the following property: there are (affine) linear polynomials $p_1, \ldots, p_m$ with integer coefficients, such that for all $x \in [0,1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x) = p_i(x)$. $\mathcal{M}_n$ is equipped with the pointwise operations $+, -, \wedge, \vee$ of $\mathbb{R}$, and with the constant function 1 as the distinguished order-unit. The characteristic universal property of $\mathcal{M}_n$ is as follows:

Proposition 2.2. ([19, 4.16]) *$\mathcal{M}_n$ is generated by the coordinate maps $\pi_i\colon [0,1]^n \to \mathbb{R}$ together with the order-unit 1. For every unital ℓ-group (G, u) and elements $g_1, \ldots, g_n$ in the unit interval $[0, u]$ of G, if $g_1, \ldots, g_n$ and u generate G, then there is a unique unital ℓ-homomorphism ψ of $\mathcal{M}_n$ onto G such that $\psi(\pi_i) = g_i$ for each $i = 1, \ldots, n$.*

We say that (G, u) is *finitely presented* if for some $n = 1, 2, \ldots$, (G, u) is isomorphic to the quotient of $\mathcal{M}_n$ by a finitely generated (= singly generated = principal) ideal.

Given $f \in \mathcal{M}_n$ we deonte $\mathcal{Z}f = a^{-1}(0)$ the *zeroset* of f. More generally, for every ideal $\mathfrak{j}$ of $\mathcal{M}_n$ we will write

$$\mathcal{Z}\mathfrak{j} = \bigcap\{\mathcal{Z}g \mid g \in \mathfrak{j}\}. \tag{1}$$

In the particular case when $\mathfrak{j}$ is maximal, $\mathcal{Z}\mathfrak{j}$ is a singleton (because the functions in $\mathcal{M}_n$ separate points, [19, 4.17]), and we write

$$\dot{\mathcal{Z}}\mathfrak{j} = \text{ the only element of } \mathcal{Z}\mathfrak{j}. \tag{2}$$

For later use we record here a classical result, whose proof follows from the Hion-Hölder theorem [8, p.45-47], [4, 2.6]:

Lemma 2.3. *For every unital ℓ-group (G, u) and ideal $\mathfrak{m} \in \mathrm{MaxSpec}\, G$ there is exactly one pair $(\iota_\mathfrak{m}, R_\mathfrak{m})$ where $R_\mathfrak{m}$ is a unital ℓ-subgroup of $(\mathbb{R}, 1)$, and $\iota_\mathfrak{m}$ is a unital ℓ-isomorphism of the quotient $(G, u)/\mathfrak{m}$ onto $R_\mathfrak{m}$. Upon identifying $(G, u)/\mathfrak{m}$ with $R_\mathfrak{m}$ every element $g/\mathfrak{m} \in (G, u)/\mathfrak{m}$ becomes a real number, and we can unambiguously write $g/\mathfrak{m} \in \mathbb{R}$.*

Corollary 2.4. *Let $\mathfrak{i}$ be an ideal of $\mathcal{M}_n$ and $\mathrm{MaxSpec}_{\supseteq \mathfrak{i}}\, \mathcal{M}_n$ denote the compact set of all maximal ideals of $\mathrm{MaxSpec}\, \mathcal{M}_n$ containing $\mathfrak{i}$. Then the map $\dot{\mathcal{Z}}$ of (2) yields a homeomorphism of $\mathrm{MaxSpec}_{\supseteq \mathfrak{i}}\, \mathcal{M}_n$ onto the compact set $\mathcal{Z}\mathfrak{i} \subseteq [0,1]^n$. The inverse of $\dot{\mathcal{Z}}$ is the map $x \in \mathcal{Z}\mathfrak{i} \mapsto \mathfrak{m}_x = \{f \in \mathcal{M}_n \mid f(x) = 0\}$, and we have identical real numbers*

$$f/\mathfrak{m} = f(\dot{\mathcal{Z}}(\mathfrak{m})), \quad \forall f \in \mathcal{M}_n, \ \forall \mathfrak{m} \in \mathrm{MaxSpec}_{\supseteq \mathfrak{i}} \in \mathcal{M}_n\,. \tag{3}$$

Proof. As a matter of fact, for each $x \in \mathcal{Z}\mathfrak{i}$, $\mathfrak{m}_x$ is a maximal ideal of $\mathcal{M}_n$. Further, for each $f \in \mathfrak{i}$, from $f(x) = 0$ we get $f \in \mathfrak{m}_x$, whence $\mathfrak{m}_x \supseteq \mathfrak{i}$ and $\dot{\mathcal{Z}}\mathfrak{m}_x = x$. Let $\mathfrak{p} \in \mathrm{MaxSpec}_{\supseteq \mathfrak{i}}\, \mathcal{M}_n$. Then $\mathcal{Z}\mathfrak{p} \subseteq \mathcal{Z}\mathfrak{i}$ and for every $f \in \mathfrak{p}$ with $f(\dot{\mathcal{Z}}\mathfrak{p}) = 0$ we have $\mathfrak{p} \subseteq \mathfrak{m}_{\dot{\mathcal{Z}}(\mathfrak{p})}$ and $\dot{\mathcal{Z}}\mathfrak{p} \in \mathcal{Z}\mathfrak{i}$. The assumed maximality of $\mathfrak{p}$ is to the effect that

$\mathfrak{p} = \mathfrak{m}_{\dot{\mathcal{Z}}(\mathfrak{p})}$, whence $\dot{\mathcal{Z}}$ is a one-one map from $\mathrm{MaxSpec}_{\supseteq\mathfrak{i}}\,\mathcal{M}_n$ onto $\mathcal{Z}\mathfrak{i}$. By definition of spectral topology, $\dot{\mathcal{Z}}$ is a homeomorphism. An application of Lemma 2.3 now settles the result. □

Corollary 2.5. *The quotient map* $\kappa\colon \mathcal{M}_n \to \mathcal{M}_n/\mathfrak{i}$ *determines the homeomorphism* $\mathfrak{m} \mapsto \mathfrak{m}/\mathfrak{i}$ *of* $\mathrm{MaxSpec}_{\supseteq\mathfrak{i}}\,\mathcal{M}_n$ *onto* $\mathrm{MaxSpec}\,\mathcal{M}_n/\mathfrak{i}$. *The inverse map is given by* $\kappa^{-1}(\mathfrak{n}) = \{f \in \mathcal{M}_n \mid f/\mathfrak{i} \in \mathfrak{n}\}$ *for each* $\mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n/\mathfrak{i}$.

Proof. The routine proof follows by combining Lemma 2.3 with [4, 2.3.8]. □

2.2. **Rational polyhedra and unimodular triangulations.** We will make use of a few elementary notions and techniques of polyhedral topology. We refer to the first chapters of [7] for background. By a *rational polyhedron* P in $\mathbb{R}^n$ we understand a finite union of simplexes $P = S_1 \cup \cdots \cup S_t$ in $\mathbb{R}^n$ such that the coordinates of the vertices of every simplex S_i are rational numbers. For every simplicial complex Σ the point-set union of the simplexes of Σ is called the *support* of Σ and is denoted $|\Sigma|$; Σ is said to be a *triangulation* of $|\Sigma|$.

For any rational point $v \in \mathbb{R}^n$ the least common denominator of the coordinates of v is called the *denominator* of v, denoted $\mathrm{den}(v)$. The integer vector $\tilde{v} = \mathrm{den}(v)(v,1) \in \mathbb{Z}^{n+1}$ is called the *homogeneous correspondent* of v. An m-simplex $U = \mathrm{conv}(w_0,\dots,w_m) \subseteq [0,1]^n$ is said to be *unimodular* if it is rational and the set of integer vectors $\{\tilde{w}_0,\dots,\tilde{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$. A simplicial complex is said to be a *unimodular triangulation* (of its support) if all its simplexes are unimodular.

As a remainder of the relevance of unimodular triangulations, recall that the homogeneous correspondent of a unimodular triangulation is known as a regular (or, nonsingular) fan [7, 22].

The following results show the connection among rational polyhedra zero-sets of McNaughton maps and ideals in $\mathcal{M}_n$.

Proposition 2.6. [18, 4.1,5.1] *Letting* $P \subseteq [0,1]^n$, *the following are equivalent:*

(i) P *is a rational polyhedron.*
(ii) $P = |\Delta|$ *for some unimodular triangulation* Δ.
(iii) *there is unimodular triangulation* ∇ *of* $[0,1]^n$ *such that*
$$P = \bigcup\{S \in \nabla\colon S \subseteq P\}.$$
(iv) $P = \mathcal{Z}f$ *for some* $f \in \mathcal{M}_n$.

Lemma 2.7. *Let* $\mathfrak{i}$ *be an ideal of* $\mathcal{M}_n$. *Then the following are equivalent:*

(i) $\mathfrak{i}$ *is principal.*
(ii) *there exists* $f \in \mathfrak{i}$ *such that* $\mathcal{Z}\mathfrak{i} = \mathcal{Z}f$.

Proof. For the non trivial direction, let $f \in \mathfrak{i}$ such that $\mathcal{Z}\mathfrak{i} = \mathcal{Z}f$. It is no loss of generality to suppose $0 \le f$. We must verify that, for all $0 \le g \in \mathcal{M}_n$, $g \in \mathfrak{i} \Leftrightarrow g \le kf$ for some $k = 1,2,\dots$. The $\leftarrow$-direction follows from the fact that $f \in \mathfrak{i}$. For the $\rightarrow$-direction, let Λ, be a rational triangulation of $[0,1]^n$, f and g are linear over each $S \in \Lambda$. Let $\{v_1,\dots,v_s\}$ be the vertices of Λ. Since $\mathcal{Z}f = \mathcal{Z}\mathfrak{i} \subseteq \mathcal{Z}g$, $f(v_i) = 0$ implies $g(v_i) = 0$. Then there exists an integer $m_i > 0$ such that $m_i f(v_i) \ge f(v_i)$ for each $i = 1,\dots,s$. Letting $m = \max(m_1,\dots,m_s)$, the desired result follows from the linearity of f and g over each simplex of Λ. □

3. Finitely presented unital ℓ-groups and basis

Theorem 3.1. *A unital ℓ-group (G, u) is finitely presented iff it has a basis.*

The $(\Rightarrow)$-direction of Theorem 3.1 is proved in [18, 5.2]. To prove the $(\Leftarrow)$-direction let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of (G, u), with multiplicities $m_1, \ldots, m_n$. Let $\kappa\colon \mathcal{M}_n \to (G, u)$ be the unique unital ℓ-homomorphism extending the map $\pi_i \mapsto b_i$, as given by Proposition 2.2. Let the ideal $\mathfrak{i}$ of $\mathcal{M}_n$ be defined by $\mathfrak{i} = \ker(\kappa)$. By Definition 2.1(i), κ is onto G, thus

$$(G, u) \cong \mathcal{M}_n \,/\mathfrak{i}. \tag{4}$$

For any $E \subseteq \mathcal{B}$ we define the simplex $T_E \subseteq [0, 1]^n$ by

$$T_E = \operatorname{conv}\{e_i/m_i \mid b_i \in E\}, \tag{5}$$

where e_i is the ith standard basis vector of $\mathbb{R}^n$. From Definition 2.1(ii) it follows that $\kappa(\sum_i m_i\pi_i) = \sum_i m_i\kappa(\pi_i) = \sum_i m_i b_i = u$ whence, defining the function $a \in \mathcal{M}_n$ by $a = |1 - \sum_i m_i\pi_i|$,

$$0 \leq a \in \mathfrak{i}, \text{ and } \mathcal{Z}a = T_{\mathcal{B}}. \tag{6}$$

Let $k = 1, 2, \ldots$. Then by a *k-cluster* of $\mathcal{B}$ we understand a k-element subset C of $\mathcal{B}$ such that $\bigwedge C \neq 0$. We denote by $\mathcal{B}^{\bowtie}$ the set of all clusters of $\mathcal{B}$. For each $C \in \mathcal{B}^{\bowtie}$, displaying the complementary set $\mathcal{B} \setminus C$ as $\{b_{j_1}, \ldots, b_{j_s}\}$, we define the function $a_C \in \mathcal{M}_n$ by

$$a_C = \pi_{j_1} \vee \ldots \vee \pi_{j_s}, \ \ (a_C = 0 \text{ in case } C = \mathfrak{B}). \tag{7}$$

We then have

$$T_{\mathcal{B}} \cap \mathcal{Z}a_C = T_C. \tag{8}$$

We next observe

$$\bigwedge_{C \in \mathcal{B}^{\bowtie}} a_C \in \mathfrak{i}. \tag{9}$$

By (7), the result is trivial if $\mathcal{B}$ is a cluster in $\mathcal{B}^{\bowtie}$. If this is not the case, let $b_{i_C} \in \mathcal{B} \setminus C$ for each $C \in \mathcal{B}^{\bowtie}$. If $D = \{b_{i_C} : C \in \mathcal{B}^{\bowtie}\} \in \mathcal{B}^{\bowtie}$, then $b_{i_D} \in D$, which is a contradiction. Therefore,

$$\kappa(\bigwedge_{C \in \mathcal{B}^{\bowtie}} \pi_{i_C}) = \bigwedge_{C \in \mathcal{B}^{\bowtie}} b_{i_C} = 0,$$

i.e., $\bigwedge_{C \in \mathcal{B}^{\bowtie}} \pi_{i_C} \in \mathfrak{i}$. Since each $b_{i_C} \in \mathcal{B} \setminus C$ is arbitrary, (9) now follows from the distributivity of the underlying lattice of (G, u).

Let the function $f^* \in \mathcal{M}_n$ be defined by

$$f^* = a \vee \bigwedge_{C \in \mathcal{B}^{\bowtie}} a_C. \tag{10}$$

From (6) and (9) it follows that

$$0 \leq f^* \in \mathfrak{i}, \tag{11}$$

and from (8),

$$\mathcal{Z}f^* = \mathcal{Z}a \ \cap \bigcup_{C \in \mathcal{B}^{\bowtie}} \mathcal{Z}a_C = \bigcup_{C \in \mathcal{B}^{\bowtie}} T_C. \tag{12}$$

From (11) we immediately have

$$\mathcal{Z}f^* \supseteq \mathcal{Z}\mathfrak{i}. \tag{13}$$

To prove the converse inclusion, for each cluster K of $\mathcal{B}$ we set

$$\mathrm{apo}(K) = \{\mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i} \mid \mathfrak{n} \supseteq \mathcal{B} \setminus K\}. \tag{14}$$

For each $\mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i}$, letting $C_\mathfrak{n}$ be the cluster of all $b \in \mathcal{B}$ such that $b \notin \mathfrak{n}$, it follows that $\mathcal{B} \setminus C_\mathfrak{n} \subseteq \mathfrak{n}$, whence $\mathfrak{n} \in \mathrm{apo}(C_\mathfrak{n})$. Thus, $\bigcup_{C\in\mathcal{B}^{\bowtie}} \mathrm{apo}(C) \supseteq \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i}$. Since the converse inclusion holds by definition, we have

$$\mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i} = \bigcup_{C\in\mathcal{B}^{\bowtie}} \mathrm{apo}(C). \tag{15}$$

For each $K \in \mathcal{B}^{\bowtie}$, we denote by $\mathrm{apo}_\mathbb{R}(K)$ the inverse image of $\mathrm{apo}(K)$ under the composition of the homeomorphisms $x \mapsto \mathfrak{m}_x \mapsto \mathfrak{m}_x/\mathfrak{i}$ of Corollaries 2.4 and 2.5, where $m_x = \{f \in \mathcal{M}_n \mid f(x) = 0\}$. In other words,

$$\mathrm{apo}_\mathbb{R}(K) = \{x \in \mathcal{Z}\mathfrak{i} \mid \mathfrak{m}_x/\mathfrak{i} \in \mathrm{apo}(K)\}. \tag{16}$$

From (12)-(15) we get

$$\bigcup_{C\in\mathcal{B}^{\bowtie}} \mathrm{apo}_\mathbb{R}(C) = \mathcal{Z}\mathfrak{i} \subseteq \mathcal{Z}f^* = \bigcup_{C\in\mathcal{B}^{\bowtie}} T_C. \tag{17}$$

This inclusion can be refined as follows:

Claim 1: For each cluster C of $\mathcal{B}$, $\mathrm{apo}_\mathbb{R}(C) \subseteq T_C$.

As a matter of fact, by (14) and condition (iii) of Definition 2.1 we have

$$\begin{aligned} \mathrm{apo}(C) &= \{\mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i} \mid b/\mathfrak{n} = 0, \forall b \in \mathcal{B} \setminus C\} \\ &= \{\mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i} \mid \frac{m_{i_1}b_{i_1} + \cdots + m_{i_t}b_{i_t}}{\mathfrak{n}} = 1\}, \end{aligned} \tag{18}$$

for each cluster $C = \{b_{i_1}, \ldots, b_{i_t}\}$ of $\mathcal{B}$. By Lemma 2.3, for each $\mathfrak{m} \in \mathrm{MaxSpec}_{\supseteq\mathfrak{i}}\,\mathcal{M}_n$ the unital ℓ-group $\frac{\mathcal{M}_n}{\mathfrak{m}}$ and its isomorphic copy $\frac{\mathcal{M}_n\,/\mathfrak{i}}{\mathfrak{m}/\mathfrak{i}}$ are canonically isomorphic to the same unital ℓ-subgroup of $(\mathbb{R}, 1)$. Thus for each $f \in \mathcal{M}_n$ we have identical real numbers $\frac{f/\mathfrak{i}}{\mathfrak{m}/\mathfrak{i}} = \frac{f}{\mathfrak{m}}$. Thus, by Corollary 2.4 and Corollary 2.5

$$\frac{f/\mathfrak{i}}{\mathfrak{n}/\mathfrak{i}} = f(\dot{\mathcal{Z}}(\kappa^{-1}(\mathfrak{n}))), \ \ \forall \mathfrak{n} \in \mathrm{MaxSpec}\,\mathcal{M}_n\,/\mathfrak{i}, \tag{19}$$

or equivalently,

$$f(x) = \frac{f/\mathfrak{i}}{\mathfrak{m}_x/\mathfrak{i}}, \ \ \forall x \in \mathcal{Z}\mathfrak{i}. \tag{20}$$

Combining (16) with (18), we obtain $(y_1, \ldots, y_n) \in \mathrm{apo}_\mathbb{R}(C)$ if and only if

$$\frac{m_{i_1}b_{i_1} + \cdots + m_{i_t}b_{i_t}}{\mathfrak{m}_x/\mathfrak{i}} = \frac{(m_{i_1}y_{i_1} + \cdots + m_{i_t}y_{i_t})/\mathfrak{i}}{\mathfrak{m}_x/\mathfrak{i}} = 1.$$

Now recalling (5), by (20) we obtain

$$\mathrm{apo}_\mathbb{R}(C) = \{(y_1, \ldots, y_n) \in \mathcal{Z}\mathfrak{i} \mid m_{i_1}y_{i_1} + \cdots + m_{i_t}y_{i_t} = 1\} \subseteq T_C, \tag{21}$$

thus settling Claim 1.

Actually, a stronger result holds:

Claim 2. For every cluster C of $\mathcal{B}$, $\mathrm{apo}_\mathbb{R}(C) = T_C$.

The proof is by induction on the number l of elements of C.

Base case: $l = 1$. Then for a unique $j \in \{1, \ldots, n\}$ we have $C = \{b_j\} = \{\pi_j/\mathfrak{i}\}$. Condition (ii) in Definition 2.1 is to the effect that $\mathrm{apo}(C)$ contains exactly one element $\mathfrak{n}$. By Lemma 2.3, $\mathfrak{n}$ is the only maximal ideal of $\mathcal{M}_n/\mathfrak{i}$ such that $0 = b/\mathfrak{n}$ for all $b \neq b_j$; by (18), $\mathfrak{n}$ is uniquely determined by the condition $1 = m_j b_j/\mathfrak{n} = (m_j\pi_j/\mathfrak{i})/\mathfrak{n}$. Letting $z \in \mathcal{Z}\mathfrak{i}$ be the image of $\mathfrak{n}$ in $\mathrm{apo}_{\mathbb{R}}(C)$, by (5) and Claim 1 we have $z = e_j/m_j$. We conclude that $\mathrm{apo}_{\mathbb{R}}(C) = \{e_j/m_j\} = \mathrm{conv}\{e_j/m_j\} = T_C$.

Induction Step, $l+1$. Let us write $C = \{b_{i_0}, \ldots, b_{i_l}\}$. Since every l-element subset C' of C is a cluster of $\mathcal{B}$, by induction hypothesis $\mathrm{apo}_{\mathbb{R}}(C') = T_{C'}$. $T_{C'}$ is known as a *facet* of T_C. By Claim 1, $\mathrm{apo}_{\mathbb{R}}(C)$ is a nonempty subset of T_C containing all facets of T_C. Further, $\mathrm{apo}_{\mathbb{R}}(C)$ is homeomorphic to an l-simplex, because so is its homeomorphic copy $\mathrm{apo}(C)$, by condition (ii) in Definition 2.1. Observe that T_C is *contractible* (i.e., T_C is continuously shrinkable to a point). By way of contradiction, suppose $\mathrm{apo}_{\mathbb{R}}(C)$ is a proper subset of T_C. Then a classical result in algebraic topology shows that $\mathrm{apo}_{\mathbb{R}}(C)$ is not contractible. Thus $\mathrm{apo}_{\mathbb{R}}(C)$ is not homeomorphic to any l-simplex, a contradiction showing $\mathrm{apo}_{\mathbb{R}}(C) = T_C$, and settling Claim 2.

Combining Claim 2 and (17), we can write

$$\mathcal{Z}f^* = \bigcup_{C \in \mathcal{B}^{\bowtie}} T_C = \mathcal{Z}\mathfrak{i}. \tag{22}$$

Recalling Lemma 2.7 it follows that $\mathfrak{i}$ is the ideal generated by f^*. By (4), (G, u) is finitely presented. The proof of Theorem 3.1 is thus complete. □

4. A class of projective unital ℓ-groups

In Theorem 4.2 below we will construct a large class of projective unital ℓ-groups. For the proof we prepare

Lemma 4.1. *Let $S = \mathrm{conv}(x_1, \ldots, x_k) \subseteq [0,1]^n$ be a unimodular $(k-1)$-simplex and $v \in \{0,1\}^n$ a vertex of $[0,1]^n$. Then for every $Y \subseteq \{x_1, \ldots, x_k\}$ there is a matrix $M \in \mathbb{Z}^{n\times n}$ and a vector $b \in \mathbb{Z}^n$ such that*

$$Mx_i + b_i = \begin{cases} v & \text{if } x_i \in Y, \\ x_i & \text{otherwise.} \end{cases} \tag{23}$$

Proof. Since S is unimodular, the set $\{\tilde{x_1}, \ldots, \tilde{x_k}\}$ of homogeneous correspondents of $x_1, \ldots, x_k$ can be extended to a basis $\{\tilde{x_1}, \ldots, \tilde{x_k}, q_{k+1}, \ldots, q_{n+1}\}$ of the free abelian group $\mathbb{Z}^{n+1}$. The $(n+1)\times(n+1)$ matrix D with column vectors $\tilde{x_1}, \ldots, \tilde{x_k}, q_{k+1}, \ldots, q_{n+1}$ is invertible and $D^{-1} \in \mathbb{Z}^{(n+1)\times(n+1)}$. For each $i = 1, \ldots, t$ let $c_i \in \mathbb{Z}^{n+1}$ be defined by

$$c_i = \begin{cases} \mathrm{den}(x_i)(v,1) & \text{if } x_i \in Y, \\ \tilde{x_i} & \text{otherwise.} \end{cases}$$

Let $C \in \mathbb{Z}^{(n+1)\times(n+1)}$ be the matrix whose columns are given by the vectors $c_1, \ldots, c_k, q_{k+1}, \ldots, q_{n+1}$. Since D and C have the same $(n+1)$th row,

$$CD^{-1} = \left(\begin{array}{c|c} M & d \\ \hline 0, \ldots, 0 & 1 \end{array}\right)$$

for some $n \times n$ integer matrix M and vector $d \in \mathbb{Z}^n$. For each $i = 1, \ldots, k$ we then have $(CD^{-1})\tilde{x_i} = (CD^{-1})\operatorname{den}(x_i)(x_i, 1) = \operatorname{den}(x_i)(Mx_i + d, 1)$. By definition, $(CD^{-1})\tilde{x_i} = c_i = \operatorname{den}(x_i)(v, 1)$ if $x_i \in Y$ and $(CD^{-1})\tilde{x_i} = \tilde{x_k} = \operatorname{den}(x_i)(x_i, 1)$ otherwise. Thus (23) is satisfied. □

Theorem 4.2. *Suppose the unital ℓ-group (G, u) has a basis $\mathcal{B}$ with $\bigwedge \mathcal{B} \neq 0$. Suppose at least one of the multiplicities of $\mathcal{B}$ is equal to 1. Then (G, u) is projective.*

Proof. By assumption, $\mathcal{B}$ itself is a basis of (G, u) with multiplicities $1 = m_1 \leq m_2 \leq \ldots \leq m_n$. We keep the notation of the proof of Theorem 3.1. In particular, $T_\mathcal{B} = \operatorname{conv}(e_1, e_2/m_2, \ldots, e_n/m_n)$, where, as the reader will recall, e_i denotes the ith basis vector in $\mathbb{R}^n$. Proposition 2.6 yields a unimodular triangulation Δ of $[0, 1]^n$ such that $T_\mathcal{B}$ is a union of simplexes of Δ, and all vertices of (every simplex of) Δ have rational coordinates.

We next define the function $\mathbf{f}\colon [0, 1]^n \to [0, 1]^n$ by stipulating that, for each vertex v of Δ,

$$\mathbf{f}(v) = \begin{cases} v & \text{if } v \in T_\mathcal{B} \\ e_1 & \text{if } v \notin T_\mathcal{B} \end{cases} \tag{24}$$

and $\mathbf{f}$ is linear over each simplex of Δ. Then $\mathbf{f}$ is a continuous map and $\mathbf{f}\restriction T_\mathcal{B}$ is the identity map on $T_\mathcal{B}$. For any simplex S of Δ, let ∂S denote the set of extremal points of S. Since $\mathbf{f}$ is linear over S and $\mathbf{f}(v) \in T_\mathcal{B}$ for each $v \in \partial S$, we have $\mathbf{f}(S) = \mathbf{f}(\operatorname{conv}(\partial S)) = \operatorname{conv}(\mathbf{f}(\partial S)) \subseteq \operatorname{conv}(T_\mathcal{B}) = T_\mathcal{B}$, whence

$$\mathbf{f}([0, 1]^n) = T_\mathcal{B}. \tag{25}$$

We have thus shown that $\mathbf{f} \circ \mathbf{f} = \mathbf{f}$ and $\mathbf{f}$ is a continuous retraction of $[0, 1]^n$ onto $T_\mathcal{B}$ which is linear on each simplex of Δ.

By Lemma 4.1, the coefficients of each linear piece of $\mathbf{f}$ are integers. Therefore, the map $\varphi\colon \mathcal{M}_n \to \mathcal{M}_n$ given by

$$\varphi(g) = g \circ \mathbf{f}. \tag{26}$$

is well defined. It follows straightforwardly that φ is a unital ℓ-homomorphism. Since $\mathbf{f} \circ \mathbf{f} = \mathbf{f}$ then $\varphi \circ \varphi = \varphi$. In other words, φ is an idempotent endomorphism of $\mathcal{M}_n$. Stated otherwise, the unital ℓ-subgroup $\varphi(\mathcal{M}_n)$ of $\mathcal{M}_n$ is a retraction of $\mathcal{M}_n$. Applying now the universal property of $\mathcal{M}_n$, (Proposition 2.2) one sees that $\mathcal{M}_n$ is projective. A routine exercise using the fact that $\varphi(\mathcal{M}_n)$ is a retraction of $\mathcal{M}_n$ shows that $\varphi(\mathcal{M}_n)$ is projective.

To conclude the proof it is enough to show that $\varphi(\mathcal{M}_n)$ is unitally ℓ-isomorphic to (G, u). In proving the ($\Leftarrow$)-direction of Theorem 3.1 we have seen that (G, u) is unitally ℓ-isomorphic to $\mathcal{M}_n / \mathfrak{i}$, for some ideal $\mathfrak{i}$ having following characterization:

$$\mathfrak{i} = \left\{ g \in \mathcal{M}_n \mid \mathcal{Z}g \supseteq \bigcup_{C \in \mathcal{B}^{\bowtie}} T_C \right\} = \{ g \in \mathcal{M}_n \mid \mathcal{Z}g \supseteq T_\mathcal{B} \}.$$

Letting $\ker(\varphi)$ be the kernel of φ, by (25) and (26) we have

$$g \in \ker(\varphi) \Leftrightarrow g \circ \mathbf{f} = 0 \Leftrightarrow g(\mathbf{f}([0, 1]^n)) = \{0\} \Leftrightarrow g(T_\mathcal{B}) = \{0\} \Leftrightarrow \mathcal{Z}g \supseteq T_\mathcal{B} \Leftrightarrow g \in \mathfrak{i}.$$

Therefore, $(G, u) \cong \mathcal{M}_n / \mathfrak{i} = \mathcal{M}_n / \ker(\varphi) \cong \varphi(\mathcal{M}_n)$, and the proof is complete. □

5. The underlying dimension group of a unital ℓ-group with a basis

In the category $\mathcal{P}$ of *partially ordered abelian groups with order-unit*, [13, p.12] objects are pairs (G, u), where G is a partially ordered abelian group and u is an order-unit of G. A morphism $\phi\colon (G, u) \to (H, v)$ of $\mathcal{P}$ is a *unital* (i.e., unit-preserving) *positive* (in the sense that $\phi(G^+) \subseteq H^+$) homomorphism.

Following [13, p.47], by a *unital simplicial* group we understand an object of $\mathcal{P}$ that is isomorphic (in $\mathcal{P}$) to the free abelian group $\mathbb{Z}^n$ for some integer $n > 0$ equipped with the product ordering: $(x_1, \ldots, x_n) \geq 0$ iff $x_i \geq 0\ \forall i = 1, \ldots, n$.

A *unital dimension group* (G, u) is an object of $\mathcal{P}$ such that $G = G^+ - G^+$, sums of intervals are intervals, and $kg \in G^+ \Rightarrow g \in G^+$, for any $g \in G$ and integer $k > 0$. For short, G is directed, Riesz, and unperforated, [13, p.44]. In [9, §2] one can find several characterizations of the Riesz property. By Elliott classification theory [12], countable unital dimension groups are complete classifiers of AF algebras, the norm limits of ascending sequences of finite-dimensional C*-algebras, all with the same unit.

Given a unital ℓ-group (G, u) let $(G, u)_{\dim}$ denote the underlying group of (G, u) equipped with the same positive cone G^+ and order-unit u, but forgetting the lattice structure of (G, u). Then $(G, u)_{\dim}$ is a unital dimension group. Thus in particular, every unital simplicial group is a unital dimension group. Since the properties of directedness, Riesz, and unperforatedness are preserved by direct limits, then direct limits of unital simplicial groups are unital dimension groups.

The celebrated Effros-Handelman-Shen theorem [6], [13, 3.21] (also see Grillet's theorem [15, 2.1] in the light of [14, Remark 3.2]) states the converse: for every unital dimension group (G, u) we can write

$$(G, u) \cong \lim\{\phi_{ij}\colon (\mathbb{Z}^{n_i}, u_i) \to (\mathbb{Z}^{n_j}, u_j) \mid i, j \in I\}$$

for some direct system of unital simplicial groups and unital positive homomorphisms in $\mathcal{P}$. For dimension groups of the form $(G, u)_{\dim}$, with (G, u) a unital ℓ-group, Marra [17] proved that the maps ϕ_{ij} can be assumed to be 1-1.

A further simplification occurs when (G, u) has a basis: as a matter of fact, in Theorem 5.3 below we will prove that the set of bases of (G, u) is rich enough to provide a direct system of unital simplicial groups and 1-1 unital homomorphisms such that $(G, u)_{\dim}$ is the limit of this system in the category $\mathcal{P}$. To this purpose, given a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of a unital ℓ-group (G, u), we let

$$\operatorname{grp} \mathcal{B} = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$$

denote the group generated by $\mathcal{B}$ in (the underlying group of) G. Similarly,

$$\operatorname{sgr} \mathcal{B} = \mathbb{Z}_{\geq 0}\, b_1 + \cdots + \mathbb{Z}_{\geq 0}\, b_n$$

will denote the semigroup generated by $\mathcal{B}$ together with the zero element.

Assuming, as we are doing throughout the rest of this paper, that the elements of $\mathcal{B}$ are listed in some prescribed order, by definition of $\mathcal{B}$ the n-tuple of multiplicities $m_{\mathcal{B}} = (m_1, \ldots, m_n)$ is uniquely determined by the n-tuple $(b_1, \ldots, b_n)$.

Proposition 5.1. *Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of a unital ℓ-group (G, u). Let*

$$G_{\mathcal{B}} = (\operatorname{grp} \mathcal{B},\ \operatorname{sgr} \mathcal{B},\ u)$$

denote the group $\operatorname{grp}\mathcal{B}$ *equipped with the positive cone* $\operatorname{sgr}\mathcal{B}$ *and with the distinguished element* $u=\sum m_i b_i$. *Let*

$$\mathbb{Z}_{\mathcal{B}} = (\mathbb{Z}^n, (\mathbb{Z}^+)^n, m_{\mathcal{B}})$$

be the standard simplicial group of rank n, *with the* n*-tuple* $m_{\mathcal{B}}$ *as a distinguished element. Then*

(1) $\mathcal{B}$ *is a free generating set of the free abelian group* $\operatorname{grp}\mathcal{B}$ *of rank* n.

(2) $G^+ \cap \operatorname{grp}\mathcal{B} = \operatorname{sgr}\mathcal{B}$.

(3) *The map* $b_i \mapsto e_i$ *uniquely extends to an isomorphism* $\psi_{\mathcal{B}}\colon \operatorname{grp}_{\mathcal{B}} \cong \mathbb{Z}^n$.

(4) $\psi_{\mathcal{B}}$ *is in fact an isomorphism (in the category* $\mathcal{P}$*) of* $G_{\mathcal{B}}$ *onto* $\mathbb{Z}_{\mathcal{B}}$, *whence* $G_{\mathcal{B}}$ *is a unital simplicial group, called the* basic group of $\mathcal{B}$; *further,* $\mathcal{B}$ *is the set of atoms (=minimal positive nonzero elements) of* $G_{\mathcal{B}}$; *thus if* $\mathcal{B}' \neq \mathcal{B}$ *is another basis of* (G,u) *then* $G_{\mathcal{B}} \neq G_{\mathcal{B}'}$.

Proof. (1) By condition (ii) in the definition of $\mathcal{B}$, no nonzero linear combination of the elements of $\mathcal{B}$ is zero in (the $\mathbb{Z}$-module) G. It is well known that G is torsion-free. Thus $\mathcal{B}$ is a free generating set in $\operatorname{grp}\mathcal{B}$, and $\operatorname{grp}\mathcal{B}$ is free abelian of rank n.

To prove (2), suppose $g \in G^+ \cap \operatorname{grp}\mathcal{B}$, and write $g=\sum_{i=1}^n l_i b_i$ for suitable integers $l_1,\dots,l_n$. Fix now $j\in\{1,\dots,n\}$ and let $\mathfrak{n}_j$ be the only maximal ideal of G such that $b_k \in \mathfrak{n}_j$ for all $k \neq j$, as given by condition (ii) in the definition of $\mathcal{B}$. By condition (iii) we have

$$0 \leq \sum_{i=1}^n l_i b_i \Rightarrow 0 \leq \frac{\sum_{i=1}^n l_i b_i}{\mathfrak{n}_j} = \frac{l_j b_j}{\mathfrak{n}_j} = \frac{l_j}{m_j},$$

whence $0 \leq l_j$ for all j, and $g \in \operatorname{sgr}\mathcal{B}$. The converse inclusion is trivial.

To prove (3) it is enough to note that the map $b_i \mapsto e_i$ is a one-one correspondence between the free generating set $\mathcal{B}$ of $\operatorname{grp}\mathcal{B}$ and the free generating set $\{e_1,\dots,e_n\}$ of $\mathbb{Z}^n$.

Concerning (4). It is easy to see that $\mathcal{B}$ is the set of atoms of $G_{\mathcal{B}}$, and $\{e_1,\dots,e_n\}$ is the set of atoms of the simplicial group $(\mathbb{Z}^n,(\mathbb{Z}^+)^n)$. Thus $\psi_{\mathcal{B}}$ is an isomorphism of $G_{\mathcal{B}}$ onto $(\mathbb{Z}^n, \mathbb{Z}^{+n})$, and $G_{\mathcal{B}}$ is simplicial. Trivially, $\psi_{\mathcal{B}}$ preserves the order-unit. So $G_{\mathcal{B}}$ is a unital simplicial group which is isomorphic (in $\mathcal{P}$) to $\mathbb{Z}_{\mathcal{B}}$. The rest is clear. □

Given two bases $\mathcal{B}'$ and $\mathcal{B}$ of a unital ℓ-group (G,u) we say that $\mathcal{B}'$ *refines* $\mathcal{B}$ if $\mathcal{B} \subseteq \operatorname{sgr}\mathcal{B}'$.

From the foregoing proposition we immediately obtain.

Proposition 5.2. *Let* $\mathcal{B}'=\{b'_1,\dots,b'_{n'}\}$ *and* $\mathcal{B}=\{b_1,\dots,b_n\}$ *be bases of a unital* ℓ*-group* (G,u) *such that* $\mathcal{B}'$ *refines* $\mathcal{B}$. *We then have:*

(1) *For each* $i=1,\dots,n$, *the element* b_i *is expressible as a linear combination* $b_i = m_{1i}b'_1+\cdots+m_{n'i}b'_{n'}$, *for uniquely determined integers* $m_{ki}\geq 0$, $(k = 1,\dots,n')$.

(2) *The* $n' \times n$ *integer matrix* $M_{\mathcal{B}\mathcal{B}'}$ *whose entries are the* m_{ki}, *has rank equal to* n.

(3) *The inclusion map* $G_{\mathcal{B}} \to G_{\mathcal{B}'}$ *induces the unital positive* 1-1 *homomorphism*

$$\phi_{\mathcal{B}\mathcal{B}'} \colon (y_1, \dots, y_n) \in \mathbb{Z}^n \mapsto (z_1, \dots, z_{n'}) = M_{\mathcal{B}\mathcal{B}'}(y_1, \dots, y_n) \in \mathbb{Z}^{n'}$$

of $(\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}})$ *into* $(\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})$*, and we have a commutative diagram*

$$\begin{array}{ccc} G_{\mathcal{B}} & \xrightarrow{\text{inclusion}} & G_{\mathcal{B}'} \\ \Big\downarrow \psi_{\mathcal{B}} & & \Big\downarrow \psi_{\mathcal{B}'} \\ (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) & \xrightarrow{\phi_{\mathcal{B}\mathcal{B}'}} & (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'}) \end{array} \tag{27}$$

Theorem 5.3. *Suppose the unital ℓ-group (G, u) has a basis. We then have:*

(1) *Any two basic groups $G_{\mathcal{B}}, G_{\mathcal{F}}$ of (G, u) are jointly embeddable (by unit preserving, order preserving inclusions) into some basic group $G_{\mathcal{B}'}$ of (G, u).*

(2) *We then have a direct system $\{\phi_{\mathcal{B}\mathcal{B}'} \colon (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) \to (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})\}$ of unital simplicial groups and unital positive* 1-1 *homomorphisms in $\mathcal{P}$, indexed by all pairs $\mathcal{B}, \mathcal{B}'$ of bases of (G, u) such that $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$.*

(3) *Further,* $\lim \{\phi_{\mathcal{B}\mathcal{B}'} \colon (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) \to (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})\} \cong (G, u)_{\dim}$.

Proof. (1) By Theorem 3.1, (G, u) is finitely presented, and for some $n = 1, 2, \dots$ we have

$$(G, u) \cong \mathcal{M}_n / \mathfrak{j}, \quad \text{for some principal ideal } \mathfrak{j} \text{ of } \mathcal{M}_n\,. \tag{28}$$

Suppose $\mathfrak{j}$ is generated by $f \in \mathcal{M}_n$. Recalling the notation $\mathcal{Z}f$ for the zeroset of f, a variant of [10, 5.2] shows that $\mathcal{M}_n / \mathfrak{j} \cong \mathcal{M}_n \restriction \mathcal{Z}f$. A fortiori, (G, u) is archimedean. From the abstract De Concini-Procesi lemma [18, 5.4] it follows that $\mathcal{B}$ and $\mathcal{F}$ have a joint refinement $\mathcal{B}'$. Direct inspection of the proof therein, shows that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by finitely many applications of the following operation: replace a 2-cluster $\{b, c\}$ of a basis $\mathcal{A}$, by the three elements $b \wedge c, b - (b \wedge c), c - (b \wedge c)$. The result is a basis $\mathcal{A}'$ such that $\mathcal{A} \subseteq \operatorname{sgr} \mathcal{A}'$. Thus $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$. The desired conclusion now follows from Proposition 5.2.

The proof of (2) now immediately follows from Proposition 5.2.

Concerning (3), in view of (27) it is sufficient to prove that $G = \bigcup\{\operatorname{grp} \mathcal{B} \mid \mathcal{B} \text{ a basis of } (G, u)\}$ and that $G^+ = \bigcup\{\operatorname{sgr} \mathcal{B} \mid \mathcal{B} \text{ a basis of } (G, u)\}$. Since $G = G^+ - G^+$, only the latter identity must be proved. In other words, we must prove:

$$\text{For every } p \in G^+,\ (G, u) \text{ has a basis } \mathcal{B} \text{ such that } p \in \operatorname{sgr} \mathcal{B}. \tag{29}$$

As remarked above, we have a unital ℓ-isomorphism $\omega \colon (G, u) \cong \mathcal{M}_n \restriction \mathcal{Z}f$. By [18, 4.5], ω induces a 1-1 correspondence between bases of the archimedean unital ℓ-group (G, u) and Schauder bases $\mathcal{H}_\Delta$ of $\mathcal{M}_n \restriction \mathcal{Z}f$, where Δ ranges over unimodular triangulations of the rational polyhedron $\mathcal{Z}f$. Trivially, $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$ iff $\omega(\mathcal{B}) \subseteq \omega(\mathcal{B}')$. Thus (29) boils down to proving that for every $0 \leq g \in \mathcal{M}_n \restriction \mathcal{Z}f$ there is a unimodular triangulation Δ of $\mathcal{Z}f$ such that $g \in \operatorname{sgr} \mathcal{H}_\Delta$. Let Δ be a unimodular triangulation of $\mathcal{Z}f$ such that g is linear over every simplex of Δ. The existence of Δ is ensured by [20, 1.2]. Since every linear piece of g has integer coefficients, for each vertex v of Δ we can write $g(v) = n_v / \operatorname{den}(v)$ for some $0 \leq n_v \in \mathbb{Z}$. As in the final part of the proof of Theorem 3.1, let $h_v \colon |\Delta| \to \mathbb{R}$ denote the Schauder hat of Δ at v. Let the function $\overline{g} \in \operatorname{sgr} \mathcal{H}_\Delta \subseteq \mathcal{M}_n \restriction \mathcal{Z}f$ be defined by

$$\overline{g} = \sum \{n_v h_v \mid v \text{ a vertex of } \Delta\}.$$

Then $\overline{g}(v) = g(v)$ for each vertex v of Δ and $\overline{g}$ is linear over each simplex of Δ. It follows that $\overline{g} = g$, which completes the proof. □

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